Weak error expansion of the implicit Euler scheme

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Abstract. In this paper, we extend the Talay Tubaro theorem to the implicit Euler scheme.

1. Introduction

Let $(\Omega, \mathcal{F}, P)$ a probability space and $T > 0$ a fixed time. W will be a Brownian motion in $\mathbf{R}$ with respect to his own filtration $\mathcal{F}_t$. We will consider the following stochastic differential equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{1.1}$$

where $x \in \mathbf{R}$, b and σ are real functions defined on $\mathbf{R}$. It is well know that, under Lipschitz conditions on b and σ, this equation admits a unique strong solution.

For various reasons, including mathematical finance or partial differential equations, the approximation of $Ef(X_T)$ is of importance. One way to do this is to use an Euler scheme and to study the speed of convergence. There is a vast literature on this subject and one of the pioneering work is the paper of D. Talay and L. Tubaro [7].

Let $N \in \mathbf{N}^*$ and $h := T/N$. Consider $(t_k)_{0\le k\le N}$ the uniform subdivision of $[0,T]$ defined by $t_k := kh$. In their paper [7] the authors deal with the explicit Euler scheme $\left(\bar{X}_{t_k}\right)_{0\le k\le N}$ defined as: $\bar{X}_{t_0} = x$ and for $k = 0, \dots, N-1$,

$$\bar{X}_{t_{k+1}} = \bar{X}_{t_k} + b\left(\bar{X}_{t_k}\right)h + \sigma\left(\bar{X}_{t_k}\right)\Delta W_{k+1}, \tag{1.2}$$

where $\Delta W_{k+1} := W_{t_{k+1}} - W_{t_k}$. They study the weak error $Ef\left(\bar{X}_T\right) - Ef\left(X_T\right)$.

Here, we will use the implicit Euler scheme defined as follow: $X^N_{t_0} = x$ and for $k = 0, \dots, N-1$,

$$X^N_{t_{k+1}} = X^N_{t_k} + b\left(X^N_{t_{k+1}}\right)h + \sigma\left(X^N_{t_k}\right)\Delta W_{k+1}. \tag{1.3}$$

Despite the fact that this implicit scheme cannot be implemented in most cases, it has been studied in [5] but, to the best of our knowledge, its weak error expansion has not been given. The main reason of this study is that we believe it would be a step in order to study a weak convergence error for SPDEs. So far in that framework only few cases have been studied in [2]-[4] for the stochastic heat or Schrödinger equation.

Notations. Let $n \in \mathbf{N}$ and $v, w : [0,T] \times \mathbf{R} \to \mathbf{R}$ be smooth functions. We will denote by $\partial^n v(t,x)$ the n^{th} derivative of v with respect to the space variable x, except for the second derivative denoted $\Delta v(t,x)$ as usual. Moreover, by an abuse of notation, for a function $v : \mathbf{R} \to \mathbf{R}$ and $w : [0,T] \times \mathbf{R} \to \mathbf{R}$, we will write $(vw)(t,x) := v(x)w(t,x)$.

Given $p \in \mathbf{N}$, C_p will denote a constant that depends on p, T and the coefficients b and σ, but does not depend on N. As usual, C_p may change from line to line.

For h small enough, we denote by S_h the functions defined on $\mathbf{R}$ by

$$S_h(x) := 1/(1 - hb'(x)). \tag{1.4}$$

It is similar to the map used by Debussche in [3].

2. The main result

Let u the (classical) solution of the following pde, called the Kolmogorov equation:

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + b(x)\partial u(t,x) + \frac{1}{2}\sigma^2(x)\Delta u(t,x) = 0, \\ u(T,x) = f(x). \end{cases} \tag{2.1}$$

The properties of u will be given in the next section. Let us mention that for b and σ smooth enough, u is smooth too. We define the function $\psi_i : [0,T] \times \mathbf{R} \to \mathbf{R}$, where i stands for implicit, as follows for a smooth enough function u:

$$\psi_i := \frac{1}{2}b\partial(b\partial u) + \frac{1}{4}\sigma^2\Delta(b\partial u) - \frac{1}{2}b^2\Delta u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{4}b\partial\left(\sigma^2\Delta u\right) - \frac{1}{8}\sigma^2\Delta\left(\sigma^2\Delta u\right). \tag{2.2}$$

We are now in position to state the main result of this paper.

Theorem 2.1. *Let b, σ, f be C^∞-functions with bounded derivatives.*
(i) The implicit Euler scheme (1.3) is of weak order 1, that is, there exists a constant C, such that for h small enough $\left|Ef\left(X_T^N\right) - Ef\left(X_T\right)\right| \leq Ch$.
(ii) The weak error can be expanted as

$$Ef\left(X_T^N\right) - Ef\left(X_T\right) = hE\int_0^T \psi_i(t,X_t)dt + O(h^2).$$

We have not given the minimal hypothesis; indeed we want to focus on the ideas and not on the best set of assumptions. The proof of this theorem is quite long; it uses intensively the Kolmogorov equation (2.1), the Itô and Clark-Ocone formulas. It will be proved in the next section. We at first compare our result with that of Talay Tubaro. In their paper [7], the authors introduce the following function

$$\psi_e = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u + \frac{1}{2}\frac{\partial^2}{\partial t^2}u + b\frac{\partial}{\partial t}\partial u + \frac{1}{2}\sigma^2\frac{\partial}{\partial t}\Delta u,$$

and prove the following result (see [7] page 489).

Theorem 2.2. *Let $(\bar{X}_{t_k})_{k=0,\ldots,N}$ denote the explicit Euler scheme defined by* (1.2)*. Then weak error has the following expansion*

$$Ef\left(\bar{X}_T\right) - Ef\left(X_T\right) = hE\int_0^T \psi_e(t,X_t)dt + O(h^2).$$

Applying $\frac{\partial}{\partial t}$, $b\partial$ and finally $\frac{1}{2}\sigma^2\Delta$ to (2.1) and summing these equations we have

$$\frac{\partial^2}{\partial t^2}u + 2b\partial\frac{\partial}{\partial t}u + \sigma^2\Delta\frac{\partial}{\partial t}u = -b\partial(b\partial u) - \frac{1}{2}b\partial\left(\sigma^2\Delta u\right) - \frac{1}{2}\sigma^2\Delta(b\partial u) - \frac{1}{4}\sigma^2\Delta\left(\sigma^2\Delta u\right)$$

So we can rewrite the function ψ_e as

$$\psi_e = \frac{1}{2}b^2\Delta u + \frac{1}{2}b\sigma^2\partial^3 u + \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{2}b\partial(b\partial u) - \frac{1}{4}b\partial\left(\sigma^2\Delta u\right) - \frac{1}{4}\sigma^2\Delta(b\partial u) - \frac{1}{8}\sigma^2\Delta\left(\sigma^2\Delta u\right)$$

For $b = 0$, we have $\psi_e = \psi_i = \frac{1}{8}\sigma^4\partial^4 u - \frac{1}{8}\sigma^2\Delta\left(\sigma^2\Delta u\right)$ as expected since in this case the explicit and the implicit Euler scheme coincide. We can notice that $\psi_i = \psi_e - b^2\Delta u + \frac{1}{2}\sigma^2\Delta(b\partial u) + b\partial(b\partial u) - \frac{1}{2}b\sigma^2\partial^3 u$.

3. Proof Theorem 2.1

Here is a sketch of the proof: After proving some property of the scheme, we introduce a continuous interpolation of this scheme. Finally, after decomposing the weak error, we study a remainder term.

3.1. Some tools.

Proposition 3.1 (Property of u)**.** *Let $\left(X_s^{t,x}\right)_{s\in[t,T]}$ denote the stochastic flow, that is the solution of* (1.1) *starting from x at time t and let $u(t,x) = Ef\left(X_T^{t,x}\right)$. Then u belongs to $C^{\infty,\infty}([0,T]\times\mathbf{R})$ and satisfies the Kolmogorov equation* (2.1)*. Moreover, for any $n,p\in\mathbf{N}$, there exists constants C and k such that*

$$\left|\frac{\partial^n}{\partial t^n}\partial^p u(t,x)\right| \leq C\left(1+|x|^k\right).$$

See for example [7] page 486 Lemma 2.

Now we recall several results from Malliavin Calculus that will be used in the sequel. For a detailled introduction, we send the reader to D.Nualart's book [6].

Proposition 3.2 (Clark-Ocone formula)**.** *Let $t\in[0,T]$ and $F\in L^2(\mathcal{F}_t)\cap D^{1,2}$; then we have for all $s\in[0,t]$*

$$F = E(F|\mathcal{F}_s) + \int_s^t E\left(D_rF|\mathcal{F}_r\right)dW_r.$$

Lemma 3.3. *Let $F,G\in D^{1,2}$.*

(i) If F and DF are bounded, then $FG\in D^{1,2}$ and $D(FG) = FDG + GDF$.

(ii) Let $f\in C^1$ with a bounded derivative; then $f(F)\in D^{1,2}$ and $Df(F) = f'(F)DF$.

(iii) Let $(s,t)\in[0,T]^2$ such that $s<t$ and let $F\in D^{1,2}\cap L^2(\mathcal{F}_s)$. Then $F(W_t-W_s)\in D^{1,2}$ and

$$D_r\left[F\left(W_t-W_s\right)\right] = D_rF(W_t-W_s) + F1_{\{s\leq r\leq t\}}.$$

(iv) Let $\{H_n, n\geq 1\}$ be a sequence of random variables in $D^{1,2}$ that converges to H in $L^2(\Omega)$ and such that $\sup_n E\left(\|DH_n\|^2_{L^2(0,T)}\right)<\infty$. Then H belongs to $D^{1,2}$.

For a proof of (iii), see [6] Lemma 1.3.4. Now we state some technical lemmas that will be useful in the sequel. The following discrete Gronwall lemma is classical.

Lemma 3.4 (Gronwall's lemma)**.** *For any nonnegative sequences $(a_k)_{0\leq k\leq N}$ and $(b_k)_{0\leq k\leq N}$ satisfying $a_{k+1}\leq(1+Ch)a_k+b_{k+1}$, with $C>0$. Then we have $a_k\leq e^{C(T-t_k)}\left(a_0+\sum_{i=1}^k b_i\right)$.*

Lemma 3.5. *Let $L>0$; then for h^* small enough (more precisely $Lh^*<1$) there exists $\Gamma := \frac{L}{1-Lh^*}>0$ such that for all $h\in(0,h^*)$ we have $\frac{1}{1-Lh}<1+\Gamma h$.*

Proof. Let $h\in(0,h^*)$; then we have $1-Lh>1-Lh^*>0$. Hence $\frac{L}{1-Lh}<\frac{L}{1-Lh^*}=\Gamma$, so that $Lh<\Gamma h(1-Lh)$, which yields $1+\Gamma h-Lh-\Gamma Lh^2=(1+\Gamma h)(1-Lh)>1$. This concludes the proof. □

Lemma 3.6 (Generalization of Young's lemma)**.** *For an integer $p\geq 1$ and for $\epsilon>0$, we have*

$$(a+b)^{2^p}\leq(1+\epsilon)^{2^p-1}a^{2^p}+\left(1+\frac{1}{\epsilon}\right)^{2^p-1}b^{2^p}.$$

Proof. We use an induction argument. The inequality is true for $p=1$, that is $(a+b)^2\leq(1+\epsilon)a^2+\left(1+\frac{1}{\epsilon}\right)b^2$. Now, suppose that it is true until p and will prove it for $p+1$; indeed the induction hypothesis yields

$$\begin{aligned}(a+b)^{2^{p+1}}\leq&\left|(1+\epsilon)^{2^p-1}a^{2^p}+\left(1+\frac{1}{\epsilon}\right)^{2^p-1}b^{2^p}\right|^2\\ \leq&(1+\epsilon)\left|(1+\epsilon)^{2^p-1}\right|^2\left|a^{2^p}\right|^2+\left(1+\frac{1}{\epsilon}\right)\left|\left(1+\frac{1}{\epsilon}\right)^{2^p-1}\right|^2\left|b^{2^p}\right|^2.\end{aligned}$$

This concludes the proof. □

3.2. **Property of the implicit Euler scheme.**

Lemma 3.7 (Existence of the scheme)**.** *For small h, the implicit Euler scheme* (1.3) *is well defined. Moreover, for all $k = 0, \dots, N$, we have $X^N_{t_k} \in L^2(\mathcal{F}_{t_k})$.*

We will denote by N_0 the smallest integer such that the scheme is well defined.

Proof. For $k = 0$, we have $X^N_{t_0} = x \in L^2(\mathcal{F}_{t_k})$. Suppose that for all $j = 0, \dots, k$, $X^N_{t_j}$ is well defined and belongs to $L^2\left(\mathcal{F}_{t_j}\right)$; we prove this for $j = k+1$. We define $\xi_{k+1} := X^N_{t_k} + \sigma\left(X^N_{t_k}\right)\Delta W_{k+1}$. By independence of ΔW_{k+1} and $\mathcal{F}_{t_k}$ and the linear growth of σ, we have that $\xi_{k+1} \in L^2(\Omega)$. Let $F_{k+1} : L^2(\Omega) \to L^2(\Omega)$ be defined by

$$F_{k+1}(X) := \xi_{k+1} + b(X)h, \tag{3.1}$$

for all $X \in L^2(\Omega)$. Using the Lipschitz property of b we have $E\left|F_{k+1}(X) - F_{k+1}(Y)\right|^2 \le \left|\|b'\|_\infty h\right|^2 E\left|X - Y\right|^2$. So by the fixed point theorem, if $\|b'\|_\infty h < 1$ there exist an unique element of $L^2(\Omega)$, noted $X^N_{t_{k+1}}$, such that $X^N_{t_{k+1}} = F_{k+1}\left(X^N_{t_{k+1}}\right)$. The measurability of $X^N_{t_{k+1}}$ with respect to $\mathcal{F}_{t_{k+1}}$ is obvious. □

Lemma 3.8 (Malliavin derivability)**.** *Let $h > 0$ small enough; then for all $k = 0, \dots, N$, we have $X^N_{t_k} \in D^{1,2}$. Moreover, for all $t \in [t_k, t_{k+1})$, we have $D_t X^N_{t_{k+1}} = S_h\left(X^N_{t_{k+1}}\right)\sigma\left(X^N_{t_k}\right)$, where S_h is defined by* (1.4)*.*

Proof. It is true for $k = 0$, since $X^N_0 = x$. Now suppose that for all $j = 1, \dots, k$, $X^N_{t_j} \in D^{1,2}$ and prove that $X^N_{t_{k+1}} \in D^{1,2}$. First, we define the following sequence in $L^2(\Omega)$: $X^N_{k+1}(0) = 0$ and for $i \ge 0$, $X^N_{k+1}(i+1) = F_{k+1}\left(X^N_{k+1}(i)\right)$ where F_{k+1} is defined by (3.1). Using the Lipschitz property of F_{k+1}, since $X^N_{t_{k+1}}$ is a fixed point of F_{k+1}, we have

$$E\left|X^N_{t_{k+1}} - X^N_{k+1}(i+1)\right|^2 \le \left|\|b'\|_\infty h\right|^2 E\left|X^N_{t_{k+1}} - X^N_{k+1}(i)\right|^2 \le \left(\left|\|b'\|_\infty h\right|^2\right)^{i+1} E\left|X^N_{t_{k+1}}\right|^2.$$

So $X^N_{k+1}(i)$ converge to $X^N_{t_{k+1}}$ in $L^2(\Omega)$ if $\|b'\|_\infty h < 1$. Using the induction hypothesis, the assumptions on σ and Lemma 3.3 (ii) and (iii), we deduce that $\xi_{k+1} = X^N_{t_k} + \sigma\left(X^N_{t_k}\right)\Delta W_{k+1}$ belongs to $D^{1,2}$. Finally, since b is Lipschitz, we deduce by induction that for all $i \ge 0$ $X^N_k(i) \in D^{1,2}$. Moreover we have $DX^N_k(i+1) = D\xi_{k+1} + hb'(X^N_k(i))DX^N_k(i)$ and $DX^N_{k+1}(0) = 0$, so that

$$\|DX^N_{t_k}(i+1)\|^2_{L^2(0,T)} \le 2\|D\xi_{k+1}\|^2_{L^2(0,T)} + 2h^2\|b'\|^2_\infty \|DX^N_{t_k}(i)\|^2_{L^2(0,T)}.$$

An induction argument yields for $i \ge 1$ and $2h^2\|b'\|^2_\infty < 1$,

$$\|DX^N_{t_k}(i)\|^2_{L^2(0,T)} \le 2\frac{\|D\xi_{k+1}\|^2_{L^2(0,T)}}{1 - 2h^2\|b'\|^2_\infty}.$$

Finally, we have $\sup_i \|DX^N_k(i)\| < \infty$. Lemma 3.3 (iv) proves that $X^N_{t_{k+1}} \in D^{1,2}$. Finally, let $t \in [t_k, t_{k+1})$; applying the Malliavin derivative D to (1.3) and using Lemma 3.3 we have

$$D_t X^N_{t_{k+1}} = hb'(X^N_{t_{k+1}})D_t X^N_{t_{k+1}} + \sigma\left(X^N_{t_k}\right);$$

which concludes the proof. □

The following result gives a bound of p^{th} moments of the implicit scheme.

Lemma 3.9. *Fix $p \geq 1$; then for N_0 large enough, there exists a constant $C(p) > 0$ such that*

$$\sup_{N \geq N_0} \max_{k=0,\ldots,N} E\left|X_{t_k}^N\right|^p \leq C(p). \tag{3.2}$$

Proof. Holder's inequality shows that it suffices to consider moments which are power of 2, that is to check $\sup_{N \geq N_0} \max_{k=0,\ldots,N} E\left|X_{t_k}^N\right|^{2^p} \leq C_p$, for every integer $p \geq 1$. Using the generalized Young Lemma 3.6 the independence between ΔW_{k+1} and $\mathcal{F}_{t_k}$, and the fact that for all $j \in \mathbf{N}$, $E\left(\Delta W_{k+1}\right)^{2j+1} = 0$, we have for $h \in (0, h^*)$ and some constant C_p depending on h^*

$$\begin{aligned}
E\left|X_{t_{k+1}}^N\right|^{2^p} \leq & (1+h)^{2^p-1} E\left|X_{t_k}^N + \sigma\left(X_{t_k}^N\right)\Delta W_{k+1}\right|^{2^p} \\
& + \left(1+\frac{1}{h}\right)^{2^p-1} h^{2^p} E\left|b\left(X_{t_{k+1}}^N\right)\right|^{2^p} \\
\leq & (1+C_p h) E\left|X_{t_k}^N\right|^{2^p} + (1+C_p h) \sum_{j=1,\ldots,2^{p-1}} \binom{2^p}{2j} E\left(\left|X_{t_k}^N\right|^{2^p-2j}\left|\sigma\left(X_{t_k}^N\right)\right|^{2j}\right) E\left|\Delta W_{k+1}\right|^{2j} \\
& + C_p h\left(1 + E\left|X_{t_{k+1}}^N\right|^{2^p}\right)
\end{aligned}$$

Using the identity $E\left|\Delta W_{k+1}\right|^{2j} = C(2j)h^j$ and the linear growth of σ we deduce for $h < 1$

$$\begin{aligned}
E\left|X_{t_{k+1}}^N\right|^{2^p} \leq & (1+C_p h) E\left|X_{t_k}^N\right|^{2^p} + C_p h + C_p h E\left|X_{t_{k+1}}^N\right|^{2^p} \\
& + (1+C_p h) C_p h \sum_{j=1,\ldots,2^{p-1}} E\left(\left|X_{t_k}^N\right|^{2^p-2j}\left(1+\left|X_{t_k}^N\right|^{2j}\right)\right).
\end{aligned}$$

Using the inequality: $a^{2^{p+1}-2j} \leq a^{2^{p+1}} + 1$ valid for any $a > 0$, we get for some constant $C_p > 0$ and $h < 1$

$$E\left|X_{t_{k+1}}^N\right|^{2^p} \leq (1+C_p h) E\left|X_{t_k}^N\right|^{2^p} + C_p h + C_p h E\left|X_{t_{k+1}}^N\right|^{2^p}$$

Provided that h is small enough, the Gronwall Lemma 3.4 and Lemma 3.5 conclude the proof. □

3.3. Some martingales and related process: $\beta_t^{k,N}, z_t^{k,N}, \gamma_t^{k,N}$ and $\eta_t^{k,N}$.

Let $k \in \{0, \ldots, N-1\}$ be fixed; in the sequel, we will use the following processes defined for $t \in [t_k, t_{k+1}]$

$$\begin{aligned}
\beta_t^{k,N} :=& E\left(b\left(X_{t_{k+1}}^N\right)\middle|\mathcal{F}_t\right), \quad z_t^{k,N} := E\left(D_t b\left(X_{t_{k+1}}^N\right)\middle|\mathcal{F}_t\right), \\
\gamma_t^{k,N} :=& \sigma\left(X_{t_k}^N\right) + (t-t_k) z_t^{k,N}, \quad \eta_t^{k,N} := \sigma\left(X_{t_k}^N\right) E\left(D_t\left(S_h b'\right)\left(X_{t_{k+1}}^N\right)\middle|\mathcal{F}_t\right).
\end{aligned} \tag{3.3}$$

The following lemma describes the time evolution of these processes.

Lemma 3.10. *For all $k = 0, \ldots, N-1$, and for $t \in [t_k, t_{k+1}]$, we have the following relation*

$$\begin{aligned}
d\beta_t^{k,N} = z_t^{k,N} dW_t, \quad dz_t^{k,N} = \eta_t^{k,N} dW_t, \quad d\gamma_t^{k,N} = z_t^{k,N} dt + \eta_t^{k,N}(t-t_k) dW_t, \\
d\eta_t^{k,N} = \left|\sigma\left(X_{t_k}^N\right)\right|^2 E\left(D_t\left(S_h^3 b''\right)\left(X_{t_{k+1}}^N\right)|\mathcal{F}_t\right) dW_t.
\end{aligned}$$

Proof. Let $k=0,\dots,N-1$ and let $t\in[t_k,t_{k+1}]$. Lemmas 3.3 (ii), 3.7 and 3.8 and the bounds of $\|b''\|_\infty$ imply that $b\left(X^N_{t_{k+1}}\right)\in L^2\left(\mathcal{F}_{t_{k+1}}\right)\cap D^{1,2}$ and

$$D_t b\left(X^N_{t_{k+1}}\right)=b'\left(X^N_{t_{k+1}}\right)D_tX^N_{t_{k+1}}=(S_hb')\left(X^N_{t_{k+1}}\right)\sigma\left(X^N_{t_k}\right). \tag{3.4}$$

So the Clark-Ocone formula in Proposition 3.2 yields $\beta^{k,N}_t=b\left(X^N_{t_{k+1}}\right)-\int_t^{t_{k+1}}z^{k,N}_s dW_s$ and hence $d\beta^{k,N}_t=z^{k,N}_t dW_t$; where $z^{k,N}_s=E\left(D_sb\left(X^N_{t_{k+1}}\right)|\mathcal{F}_s\right)$; using (3.4)

$$z^{k,N}_s=\sigma\left(X^N_{t_k}\right)E\left((S_hb')\left(X^N_{t_{k+1}}\right)\Big|\mathcal{F}_s\right). \tag{3.5}$$

So taking conditionnal expectation with respect to $\mathcal{F}_t$, we have (3.5). Since b'' is bounded and b' Lipschitz we have that for h small enough, $S_hb'=\frac{b'}{1-hb'}\in C^1_b$. So we can conclude that $(S_hb')\left(X^N_{t_{k+1}}\right)\in D^{1,2}$ and using the Clark-Ocone formula we deduce that for $s\in[t_k,t_{k+1})$,

$$(S_hb')\left(X^N_{t_{k+1}}\right)=E\left(S_hb'\left(X^N_{t_{k+1}}\right)|\mathcal{F}_s\right)+\int_s^{t_{k+1}}E\left(D_u\left[S_hb'\left(X^N_{t_{k+1}}\right)\right]|\mathcal{F}_u\right)dW_u,$$

and hence

$$dz^{k,N}_s=\sigma\left(X^N_{t_k}\right)E\left(D_s\left[\frac{b'}{1-hb'}\left(X^N_{t_{k+1}}\right)\right]\Big|\mathcal{F}_s\right)dW_s=\eta^{k,N}_s dW_s.$$

The differential of $\gamma^{k,N}_t$ is a consequence of the previous result and Itô's formula. Finally, since $S_hb'\in C^1_b$ and $(S_hb')'=S_h^2b''$, Lemma 3.8 implies that $D_t(S_hb')\left(X^N_{t_{k+1}}\right)=\sigma\left(X^N_{t_k}\right)(S^3_hb'')\left(X^N_{t_{k+1}}\right)$ and then

$$\eta^{k,N}_t=\left|\sigma\left(X^N_{t_k}\right)\right|^2E\left((S^3_hb'')\left(X^N_{t_{k+1}}\right)|\mathcal{F}_t\right). \tag{3.6}$$

Applying once more the Clark-Ocone formula in Proposition 3.2, we deduce

$$E\left(S^3_hb''\left(X^N_{t_{k+1}}\right)|\mathcal{F}_t\right)=(S^3_hb'')\left(X^N_{t_{k+1}}\right)-\int_t^{t_{k+1}}E\left(D_s(S^3_hb'')\left(X^N_{t_{k+1}}\right)|\mathcal{F}_s\right)dW_s.$$

Multiplying this by $\left|\sigma\left(X^N_{t_k}\right)\right|^2$ and using (3.6), we conclude the proof. □

The next lemma provides uniform moment estimates of the above processes.

Lemma 3.11. *Let $p\in\mathbf{N}$; then for N_0 large enough there exists a constant C_p such that for $N\ge N_0$,*

$$\max_{k=0,\dots,N-1}\sup_{t_k\le t\le t_{k+1}}E\left\{\left|\beta^{k,N}_t\right|^p+\left|z^{k,N}_t\right|^p+\left|\gamma^{k,N}_t\right|^p+\left|\eta^{k,N}_t\right|^p\right\}\le C_p.$$

Proof. Using Jensen's inequality, the Lipschitz property of b and Lemma 3.9 we have

$$E\left|\beta^{k,N}_t\right|^p\le E\left|b\left(X^N_{t_{k+1}}\right)\right|^p\le C_p\left(1+E\left|X^N_{t_{k+1}}\right|^p\right)\le C_p.$$

The identity (3.5), Jensen's inequality, the growth property of σ and the upper estimate $b'/(1-hb')\ge C$ for small h, Schwarz's inequality and Lemma 3.9 yield

$$E\left|z^{k,N}_t\right|^p\le E\left|\sigma\left(X^N_{t_k}\right)(S_hb')\left(X^N_{t_{k+1}}\right)\right|^p\le C_p.$$

Using the definition of $\gamma^{k,N}_t$ in (3.3) and the previous upper estimates we deduce

$$E\left|\gamma^{k,N}_t\right|^p\le C_pE\left|\sigma\left(X^N_{t_k}\right)\right|^p+C_ph^pE\left|z^{k,N}_t\right|^p\le C_p.$$

Finally (3.6), the Jensen inequality, the growth condition on σ, the upper bounds of b' and b'', Lemma 3.9 and Schwarz's inequality yield

$$E\left|\eta_t^{k,N}\right|^p \leq E\left[\left|\sigma\left(X_{t_k}^N\right)\right|^{2p}\left|\left(S_h^3 b''\right)\left(X_{t_{k+1}}^N\right)\right|^p\right] \leq C_p$$

This concludes the proof. □

3.4. **Continuous interpolation.** As usual we need to introduce a continuous process that interpolates the implicit Euler scheme (1.3). With an abuse of notation, let $\left(X_t^N\right)_{t\in[0,T]}$ be the process defined as follow: $X_0^N = x_0$ and for $k = 0,\ldots,N-1$ and $t_k \leq t \leq t_{k+1}$

$$X_t^N := X_{t_k}^N + E\left(b\left(X_{t_{k+1}}^N\right)\middle|\mathcal{F}_t\right)(t-t_k) + \sigma\left(X_{t_k}^N\right)\left(W_t - W_{t_k}\right). \tag{3.7}$$

This process satisfies the following

Lemma 3.12. *The process $\left(X_t^N\right)_{t\in[0,T]}$ is continuous $\mathcal{F}_t$-adapted and is an interpolation of the scheme* (1.3). *Moreover, for $k \in \{0,\ldots,N-1\}$ and $t \in [t_k, t_{k+1}]$ we have $dX_t^N = \beta_t^{k,N}dt + \gamma_t^{k,N}dW_t$, where the process $(\beta_t^{k,N})$ and $(\gamma_t^{k,N})$ are defined by* (3.3).

Remark 3.13. (1) If $b = 0$, (3.7) corresponds to the classical interpolation given by Talay Tubaro [7], since the explicit and implicit Euler scheme are the same.
(2) If b is linear, this continuous process differs from that used by Debussche in [3]. Indeed, the finite dimensional analog of the interpolation correponding to the process $dX_t = -\beta X_t dt + \sigma(X_t)\,dW_t$, is defined by

$$X_t^D = X_{t_k}^N + \int_{t_k}^t \frac{-\beta X_{t_k}^N}{1+h\beta}ds + \int_{t_k}^t \frac{\sigma\left(X_{t_k}^N\right)}{1+h\beta}dW_s;$$

for $t \in [t_k, t_{k+1}]$ (see [3] page 96 equation (3.2)). In this particular case, our interpolation is given by

$$X_t^N = X_{t_k}^N + \int_{t_k}^t E\left(-\beta X_{t_{k+1}}^N|\mathcal{F}_s\right)ds + \int_{t_k}^t \left\{\sigma\left(X_{t_k}^N\right) + (s-t_k)E\left(-D_s\beta X_{t_{k+1}}^N\middle|\mathcal{F}_s\right)\right\}dW_s.$$

Proof of Lemma 3.12. The fact that (X_t^N) is an $(\mathcal{F}_t)$-adapted process which interpolates the scheme (1.3) is a consequence of (3.7). The continuity is a consequence of the fact that the map $(t \mapsto E(X|\mathcal{F}_t))$ has a continuous modification. So, applying Itô's formula and Lemma 3.10, we obtain $d(\beta_t^{k,N}(t-t_k)) = (t-t_k)\,z_t^{k,N}dW_t + \beta_t^{k,N}dt$, and hence

$$dX_t^N = \beta_t^{k,N}dt + \left(\sigma\left(X_{t_k}^N\right) + (t-t_k)\,z_t^{k,N}\right)dW_t.$$

This concludes the proof. □

We next give moment estimates of the interpolation process X_t^N.

Lemma 3.14. *Let $p \geq 1$ and $h^* > 0$ be small enough. There exits a constant $C_p > 0$ depending on h^* such that*

$$\sup_{N\geq N_0}\sup_{t\in[0,T]} E\left|X_t^N\right|^p < C_p.$$

Proof. Using Lemma 3.9, Jensen's inequality and the independence of $W_t - W_{t_k}$ and $X_{t_k}^N$, we have for $t \in [t_k, t_{k+1}]$:

$$\begin{aligned}E\left|X_t^N\right|^p \leq & C_pE\left|X_{t_k}^N\right|^p + C_pE\left|E\left(b\left(X_{t_{k+1}}^N\right)|\mathcal{F}_t\right)\right|^p|t-t_k|^p + C_pE\left|\sigma\left(X_{t_k}^N\right)\right|^p\left|W_t - W_{t_k}\right|^p\\ \leq & C_p + C_ph^pE\left|b\left(X_{t_{k+1}}^N\right)\right|^p + E\left|\sigma\left(X_{t_k}^N\right)\right|^pE\left|W_t - W_{t_k}\right|^p.\end{aligned}$$

Using the growth condition on b and σ, moments of the normal law and Lemma 3.9, we deduce the result. □

The following is a straightforward consequence of Lemmas 3.11 and 3.14

Corollary 3.15. *Let $v : [0,T] \times \mathbf{R} \to \mathbf{R}$ be a function with polynomial growth, and let $n_1, \ldots, n_6$ non negative integers. Then there exists a constant C independent of h^* such that for $r \in [t_k, t_{k+1}]$ and $h \in (0, h^*)$*

$$E\left(\left|\beta_r^{k,N}\right|^{n_1}\left|z_r^{k,N}\right|^{n_2}\left|\gamma_r^{k,N}\right|^{n_3}\left|\eta_r^{k,N}\right|^{n_4}|r-t_k|^{n_5}\left|v\left(r,X_r^N\right)\right|^{n_6}\right) \leq C.$$

3.5. **Local decomposition.** Now we return to the proof of the main theorem. Let u be the solution to the Kolmogorov equation (2.1). Using (2.1), we decompose the weak error into a sum of local errors. Let $\delta_k^N := Eu\left(t_{k+1}, X_{t_{k+1}}^N\right) - Eu\left(t_k, X_{t_k}^N\right)$; we deduce

$$Ef\left(X_T^N\right) - Ef\left(X_T\right) = Eu\left(T, X_T^N\right) - Eu\left(0,x\right) = \sum_{k=0}^{N-1} \delta_k^N. \tag{3.8}$$

We introduce, for $t_k \leq t \leq t_{k+1}$,

$$\mathcal{I}_k^N(t) := E\left[\left(\beta_t^{k,N} - b\left(X_t^N\right)\right)\partial u\left(t, X_t^N\right)\right], \quad \mathcal{J}_k^N(t) := E\left[\left(\left|\gamma_t^{k,N}\right|^2 - \sigma^2\left(X_t^N\right)\right)\Delta u\left(t, X_t^N\right)\right].$$

Since $u \in C^{1,2}$, using Itô's formula, Lemma 3.11 and the Kolmogorov equation (2.1) at the point $\left(t, X_t^N\right)$, we obtain

$$\begin{aligned} \delta_k^N =& E\int_{t_k}^{t_{k+1}}\left\{\frac{\partial}{\partial t}u + \beta_t^{k,N}\partial u + \frac{1}{2}\left|\gamma_t^{k,N}\right|^2\Delta u\right\}\left(t, X_t^N\right)dt & (3.9)\\ =& E\int_{t_k}^{t_{k+1}}\left\{\mathcal{I}_k^N(t) + \frac{1}{2}\mathcal{J}_k^N(t)\right\}dt. & (3.10)\end{aligned}$$

Now for $k = 0, \ldots, N-1$, we introduce the following quantities for $s \in [t_k, t_{k+1}]$:

$$\begin{aligned} i_k^N(s) :=& \frac{\partial}{\partial s}(b\partial u)\left(s, X_s^N\right) + \beta_s^{k,N}\partial(b\partial u)\left(s, X_s^N\right) & (3.11)\\ &+ \frac{1}{2}\left|\gamma_s^{k,N}\right|^2\Delta(b\partial u)\left(s, X_s^N\right) - \beta_s^{k,N}\frac{\partial}{\partial s}\partial u\left(s, X_s^N\right)\\ &- \left(\left|\beta_s^{k,N}\right|^2 + z_s^{k,N}\gamma_s^{k,N}\right)\Delta u\left(s, X_s^N\right) - \frac{1}{2}\beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2\partial^3 u\left(s, X_s^N\right),\end{aligned}$$

$$\begin{aligned} j_k^N(s) :=& \left|\gamma_s^{k,N}\right|^2\frac{\partial}{\partial s}\Delta u\left(s, X_s^N\right) + \beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2\partial^3 u\left(s, X_s^N\right) & (3.12)\\ &+ \frac{1}{2}\left|\gamma_s^{k,N}\right|^4\partial^4 u\left(s, X_s^N\right) + 2\gamma_s^{k,N}z_s^{k,N}\Delta u\left(s, X_s^N\right)\\ &+ |s-t_k|^2\left|\eta_s^{k,N}\right|^2\Delta u\left(s, X_s^N\right) + 2(s-t_k)\left|\gamma_s^{k,N}\right|^2\eta_s^{k,N}\partial^3 u\left(s, X_s^N\right)\\ &- \frac{\partial}{\partial s}\left(\sigma^2\Delta u\right)\left(s, X_s^N\right) - \beta_s^{k,N}\partial\left(\sigma^2\Delta u\right)\left(s, X_s^N\right) - \frac{1}{2}\left|\gamma_s^{k,N}\right|^2\Delta\left(\sigma^2\Delta u\right)\left(s, X_s^N\right).\end{aligned}$$

The next two lemmas explain that, up to some sign, $\mathcal{I}_k^N$ (resp. $\mathcal{J}_k^N$) can be viewed as an antiderivative of i_k^N (resp. j_k^N).

Lemma 3.16. *For all $k = 0, \ldots, N-1$, we have $\mathcal{I}_k^N(t) = E\int_t^{t_{k+1}} i_k^N(s)ds$ for $t \in [t_k, t_{k+1}]$.*

Proof. If we denote by $A := E\left[\beta_t^{k,N}\partial u\left(t, X_t^N\right)\right] - E\left[b\left(X_{t_{k+1}}^N\right)\partial u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right]$ and by $B := E\left[b\left(X_{t_{k+1}}^N\right)\partial u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] - E\left[b\left(X_t^N\right)\partial u\left(t, X_t^N\right)\right]$ we can write $\mathcal{I}_k^N(t) = A + B$.

Lemma 3.11 enables us to apply Itô's formula: Let $v : [0,T] \times \mathbf{R} \to \mathbf{R}$ be of class $C^{1,2}$; Itô's formula yields

$$dv\left(t, X_t^N\right) = \left\{ \frac{\partial}{\partial t} v + \beta_t^{k,N} \partial v + \frac{1}{2} \left| \gamma_t^{k,N} \right|^2 \Delta v \right\} \left(t, X_t^N\right) dt + \gamma_t^{k,N} \partial v \left(t, X_t^N\right) dW_t. \quad (3.13)$$

Using this equation with Lemma 3.10 we have for $v \in C^{1,2}$

$$\begin{aligned} d\left[\beta_r^{k,N} v\left(r, X_r^N\right)\right] = &\left\{ \beta_r^{k,N} \frac{\partial}{\partial r} v + \left|\beta_r^{k,N}\right|^2 \partial v + \frac{1}{2} \beta_r^{k,N} \left|\gamma_r^{k,N}\right|^2 \Delta v + z_r^{k,N} \gamma_r^{k,N} \partial v \right\} \left(r, X_r^N\right) dr \\ &+ \left\{ \beta_r^{k,N} \gamma_r^{k,N} \partial v + z_r^{k,N} v \right\} \left(r, X_r^N\right) dW_r. \end{aligned} \quad (3.14)$$

The function Δu has polynomial growth; hence corollary 3.15 implies that $E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \gamma_s^{k,N} \Delta u + z_s^{k,N} \partial u \right\} \left(s, X_s^N\right) dW_s = 0$. Using equation (3.14) with $v = \partial u$, integrating between t and t_{k+1}, using the fact that $\beta_{t_{k+1}}^{k,N} = b\left(X_{t_{k+1}}^N\right)$ and taking expectation we obtain

$$A = -E \int_t^{t_{k+1}} \left\{ \beta_s^{k,N} \frac{\partial}{\partial s} \partial u + \left|\beta_s^{k,N}\right|^2 \Delta u + \frac{1}{2} \beta_s^{k,N} \left|\gamma_s^{k,N}\right|^2 \partial^3 u + z_s^{k,N} \gamma_s^{k,N} \Delta u \right\} \left(s, X_s^N\right) ds. \quad (3.15)$$

Similarly, Corollary 3.15 implies that $E \int_t^{t_{k+1}} \gamma_s^{k,N} \partial\left(b \partial u\right) \left(s, X_s^N\right) dW_s = 0$. Using (3.13) with $v = b\partial u$, integrating between t and t_{k+1} and taking expectation yields

$$B = E \int_t^{t_{k+1}} \left\{ \frac{\partial}{\partial s} \left(b \partial u\right) + \beta_s^{k,N} \partial\left(b\partial u\right) + \frac{1}{2} \left|\gamma_s^{k,N}\right|^2 \Delta\left(b\partial u\right) \right\} \left(s, X_s^N\right) ds.$$

The stochastic integral is centered by Corollary 3.15. This identity combined with (3.15) concludes the proof. □

Lemma 3.17. *For all $k = 0, \ldots, N-1$, we have $\mathcal{J}_k^N(t) = E \int_{t_k}^t j_k^N(s) ds$ for $t \in [t_k, t_{k+1}]$.*

Proof. Using (3.3) we clearly deduce that $\mathcal{J}_k^N(t) = C + D$ where

$$\begin{aligned} C :=& E\left[\left|\sigma\left(X_{t_k}^N\right) + (t - t_k) z_t^{k,N}\right|^2 \Delta u\left(t, X_t^N\right)\right] - E\left[\sigma^2\left(X_{t_k}^N\right) \Delta u\left(t_k, X_{t_k}^N\right)\right], \\ D :=& E\left[\sigma^2\left(X_{t_k}^N\right) \Delta u\left(t_k, X_{t_k}^N\right)\right] - E\left[\sigma^2\left(X_t^N\right) \Delta u\left(t, X_t^N\right)\right]. \end{aligned}$$

We at first rewrite the term D: using (3.13) with $v = \sigma^2 \Delta u$, integrating between t_k and t and taking expectation, we obtain:

$$D = -E \int_{t_k}^t \left\{ \frac{\partial}{\partial t} \left(\sigma^2 \Delta u\right) + \beta_s^{k,N} \partial\left(\sigma^2 \Delta u\right) + \frac{1}{2} \left|\gamma_s^{k,N}\right|^2 \Delta\left(\sigma^2 \Delta u\right) \right\} \left(s, X_s^N\right) ds,$$

since $\sigma^2 \Delta u$ has polynomial growth which implies that the stochastic integral is centered using Corollary 3.15. Itô's formula and Lemma 3.10 yield for $r \in [t_k, t_{k+1}]$

$$d\left|\gamma_r^{k,N}\right|^2 = \left\{ 2\gamma_r^{k,N} z_r^{k,N} + \left|\eta_r^{k,N}\right|^2 |r - t_k|^2 \right\} dr + 2\gamma_r^{k,N} \eta_r^{k,N} (r - t_k) dW_r. \quad (3.16)$$

Using this equation and (3.13), we have for v of class $C^{1,2}$ and $r \in [t_k, t_{k+1}]$

$$d\left|\gamma_r^{k,N}\right|^2 v\left(r, X_r^N\right) = \left\{\left|\gamma_r^{k,N}\right|^2 \frac{\partial}{\partial r} v + \beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2 \partial v + \frac{1}{2}\left|\gamma_r^{k,N}\right|^4 \Delta v \right. \tag{3.17}$$

$$\left. + 2\gamma_r^{k,N} z_r^{k,N} v + \left|\eta_r^{k,N}\right|^2 |r - t_k|^2 v + 2\left|\gamma_r^{k,N}\right|^2 \eta_r^{k,N}(r - t_k)\partial v\right\}\left(r, X_r^N\right) dr$$

$$+ \left\{\left(\gamma_r^{k,N}\right)^3 \partial v + 2\gamma_r^{k,N}\eta_r^{k,N}(r - t_k) v\right\}\left(r, X_r^N\right) dW_r.$$

Using equation (3.17) with $v = \Delta u$, integrating between t_k and t, using the identity $\gamma_{t_k}^{k,N} = \sigma\left(X_{t_k}^N\right)$ and taking expectation, we deduce

$$C = E\int_{t_k}^{t}\left\{\left|\gamma_s^{k,N}\right|^2 \frac{\partial}{\partial s}\Delta u + \beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2 \partial^3 u + \frac{1}{2}\left|\gamma_s^{k,N}\right|^4 \partial^4 u\right.$$

$$\left. + 2\gamma_s^{k,N} z_s^{k,N}\Delta u + |s - t_k|^2\left|\eta_s^{k,N}\right|^2 \Delta u + 2(s - t_k)\left|\gamma_s^{k,N}\right|^2 \eta_s^{k,N}\partial^3 u\right\}\left(s, X_s^N\right) ds.$$

Indeed, once more Corollary 3.15 and the polynomial growth of $\partial\Delta u$ and Δu implies that the corresponding stochastic integral is centered. This concludes the proof. □

Plugging the results of Lemmas 3.16 and 3.17 into (3.10) we obtain

$$Ef\left(X_T^N\right) - Ef\left(X_T\right) = \sum_{k=0}^{N-1} E\int_{t_k}^{t_{k+1}}\left\{\int_t^{t_{k+1}} i_k^N(s)ds + \frac{1}{2}\int_{t_k}^{t} j_k^N(s)ds\right\} dt. \tag{3.18}$$

Note: Thanks to Corollary 3.15 and the assumptions growth or boundness on the coefficients, all the stochastic integrals appearing in the next section, are centered.

3.6. **Upper estimate of $\mathcal{I}_k^N(t)$.** We next upper estimate the difference $\phi_i(s) - \phi_i(t_{k+1})$, where ϕ_i is one of the seven terms in the right hand side of (3.11)

3.6.1. *The term* $\phi_1(s) = \frac{\partial}{\partial s}(b\partial u)\left(s, X_s^N\right)$. Using (3.13) with $v = \frac{\partial}{\partial t}(b\partial u)$, integrating from s to t_{k+1} and taking expected value we deduce

$$E\frac{\partial}{\partial s}(b\partial u)\left(s, X_s^N\right) = E\frac{\partial}{\partial s}(b\partial u)\left(t_{k+1}, X_{t_{k+1}}^N\right) + R_1(s),$$

where

$$R_1(s) := -E\int_s^{t_{k+1}}\left\{\frac{\partial^2}{\partial s^2}(b\partial u) + \beta_r^{k,N}\partial\frac{\partial}{\partial s}(b\partial u) + \frac{1}{2}\left|\gamma_r^{k,N}\right|^2 \Delta\frac{\partial}{\partial s}(b\partial u)\right\}\left(r, X_r^N\right) dr.$$

Futhermore, Lemmas 3.14 and 3.11 and the polynomial growth of the functions involved imply that $|R_1(s)| \leq Ch$.

3.6.2. *The term* $\phi_2(s) = \beta_s^{k,N}\partial(b\partial u)\left(s, X_s^N\right)$. Using (3.14) with $v = \partial(b\partial u)$, integrating between s and t_{k+1} and taking expectation we obtain

$$E\left[\beta_s^{k,N}\partial(b\partial u)\left(s, X_s^N\right)\right] = E\left[b\left(X_{t_{k+1}}^N\right)\partial(b\partial u)\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_2(s),$$

where

$$R_2(s) := -E\int_s^{t_{k+1}}\left[\beta_r^{k,N}\left\{\frac{\partial}{\partial s}\partial(b\partial u) + \beta_r^{k,N}\Delta(b\partial u) + \frac{1}{2}\left|\gamma_r^{k,N}\right|^2 \partial^3(b\partial u)\right\}\right.$$

$$\left. + \gamma_r^{k,N} z_r^{k,N}\Delta(b\partial u)\right]\left(r, X_r^N\right) dr.$$

The polynomial growth of the functions and Lemmas 3.14 and 3.11 imply that $|R_2(s)| \leq Ch$.

3.6.3. *The term* $\phi_3(s) = \frac{1}{2}\left|\gamma_s^{k,N}\right|^2 \Delta\left(b\partial u\right)\left(s, X_s^N\right)$. Let

$$R_3(s) := -\frac{1}{2}E\int_s^{t_{k+1}} \left\{\left|\gamma_r^{k,N}\right|^2 \frac{\partial}{\partial t}\Delta(b\partial u) + \beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2 \partial^3(b\partial u) + \frac{1}{2}\left|\gamma_r^{k,N}\right|^4 \partial^4(b\partial u)\right.$$
$$\left. +2\gamma_r^{k,N} z_r^{k,N}\Delta(b\partial u) + \left|\eta_r^{k,N}\right|^2 |r-t_k|^2\Delta(b\partial u) + 2\left|\gamma_r^{k,N}\right|^2 \eta_r^{k,N}(r-t_k)\partial^3(b\partial u)\right\}\left(r, X_r^N\right) dr.$$

Using (3.17) with $v = \frac{1}{2}\Delta(b\partial u)$, integrating between s and t_{k+1}, and taking expectation give us

$$\frac{1}{2}E\left[\left|\gamma_s^{k,N}\right|^2 \Delta\left(b\partial u\right)\left(s, X_s^N\right)\right] = \frac{1}{2}E\left[\left|\gamma_{t_{k+1}}^{k,N}\right|^2 \Delta\left(b\partial u\right)\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_3(s),$$

with $|R_3(s)| \le Ch$.

3.6.4. *The term* $\phi_4(s) = \beta_s^{k,N}\frac{\partial}{\partial s}\partial u\left(s, X_s^N\right)$. Let

$$R_4(s) := E\int_s^{t_{k+1}} \left[\beta_r^{k,N}\frac{\partial^2}{\partial s^2}\partial u + \left|\beta_r^{k,N}\right|^2 \partial\frac{\partial}{\partial s}\partial u + \frac{1}{2}\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2 \Delta\frac{\partial}{\partial s}\partial u\right.$$
$$\left. + \gamma_r^{k,N} z_r^{k,N}\partial\frac{\partial}{\partial s}\partial u\right]\left(r, X_r^N\right) dr.$$

Using (3.14) for $v = \frac{\partial}{\partial s}\partial u$ and integrating between s and t_{k+1}, we obtain

$$-E\left[\beta_s^{k,N}\frac{\partial}{\partial t}\partial u\left(s, X_s^N\right)\right] = -E\left[b\left(X_{t_{k+1}}^N\right)\frac{\partial}{\partial t}\partial u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_4(s),$$

with $|R_4(s)| \le Ch$.

3.6.5. *The term* $\phi_5(s) = \left|\beta_s^{k,N}\right|^2 \Delta u\left(s, X_s^N\right)$. Using Itô's formula and Lemma 3.10 we have

$$d\left|\beta_r^{k,N}\right|^2 = \left|z_r^{k,N}\right|^2 dr + 2\beta_r^{k,N} z_r^{k,N} dW_r.$$

Using this equation, (3.13) and Itô's formula we obtain

$$d\left[\left|\beta_r^{k,N}\right|^2 \Delta u\left(r, X_r^N\right)\right] = \left[\left|\beta_r^{k,N}\right|^2 \frac{\partial}{\partial t}\Delta u + \left(\beta_r^{k,N}\right)^3 \partial^3 u + \frac{1}{2}\left|\beta_r^{k,N}\right|^2 \left|\gamma_r^{k,N}\right|^2 \partial^4 u\right.$$
$$\left. + \left|z_r^{k,N}\right|^2 \Delta u + 2\beta_r^{k,N} z_r^{k,N}\gamma_r^{k,N}\partial^3 u\right]\left(r, X_r^N\right) dr + dM_r,$$

where $dM_r = \left\{2\beta_r^{k,N} z_r^{k,N}\Delta u + \left|\beta_r^{k,N}\right|^2 \gamma_r^{k,N}\partial^3 u\right\}\left(r, X_r^N\right) dW_r$ and M_t is a square integrable martingale. Let

$$R_5(s) := E\int_s^{t_{k+1}} \left[\left|\beta_r^{k,N}\right|^2 \frac{\partial}{\partial r}\Delta u + \left(\beta_r^{k,N}\right)^3 \partial^3 u + \frac{1}{2}\left|\beta_r^{k,N}\right|^2 \left|\gamma_r^{k,N}\right|^2 \partial^4 u\right.$$
$$\left. + z_r^{k,N}\Delta u + 2\beta_r^{k,N} z_r^{k,N}\gamma_r^{k,N}\partial^3 u\right]\left(r, X_r^N\right) dr.$$

Integrating between s and t_{k+1} and taking expectation we have

$$-E\left[\left|\beta_s^{k,N}\right|^2 \Delta u\left(s, X_s^N\right)\right] = -E\left[\left|\beta_{t_{k+1}}^{k,N}\right|^2 \Delta u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_5(s),$$

with $|R_5(s)| \le Ch$.

3.6.6. *The term* $\phi_6(s) = z_s^{k,N}\gamma_s^{k,N}\Delta u\left(s, X_s^N\right)$. Applying Itô's formula to the product of $z_r^{k,N}\gamma_r^{k,N}$ and (3.13), and Lemma 3.10, we obtain for $r \in [t_k, t_{k+1}]$

$$d\left[z_r^{k,N}\gamma_r^{k,N} v\left(r, X_r^N\right)\right] = \left\{z_r^{k,N}\gamma_r^{k,N}\frac{\partial}{\partial t}v + z_r^{k,N}\gamma_r^{k,N}\beta_r^{k,N}\partial v + \frac{1}{2}z_r^{k,N}\left(\gamma_r^{k,N}\right)^3\Delta v + \left|z_r^{k,N}\right|^2 v\right.$$
$$\left. + \left|\eta_r^{k,N}\right|^2 (r-t_k)v + z_r^{k,N}\eta_r^{k,N}(r-t_k)\gamma_r^{k,N}\partial v + \left|\gamma_r^{k,N}\right|^2\eta_r^{k,N}\partial v\right\}\left(r, X_r^N\right)dr + dM_r \tag{3.19}$$

where $dM_r = \left\{z_r^{k,N}\left|\gamma_r^{k,N}\right|^2\partial v + z_r^{k,N}\eta_r^{k,N}(r-t_k)v + \gamma_r^{k,N}\eta_r^{k,N}v\right\}\left(r, X_r^N\right)dW_r$ and M_r is a square integrable martingale. Using equation (3.19) with $v = \Delta u$, integrating between s and t_{k+1} and taking expectation give us

$$-E\left[z_s^{k,N}\gamma_s^{k,N}\Delta u\left(s, X_s^N\right)\right] = -E\left[z_{t_{k+1}}\gamma_{t_{k+1}}^{k,N}\Delta u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_6(s),$$

with $|R_6(s)| \le Ch$.

3.6.7. *The term* $\phi_7(s) = \frac{1}{2}\beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2\partial^3 u\left(s, X_s^N\right)$. Using Lemma 3.10 and equation (3.17), Itô's formula give us for v of class $C^{1,2}$

$$d\left[\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2 v\right]\left(r, X_r^N\right) = \left\{\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2\frac{\partial}{\partial r}v + \left|\beta_r^{k,N}\right|^2\left|\gamma_r^{k,N}\right|^2\partial v + \frac{1}{2}\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^4\Delta v\right. \tag{3.20}$$
$$+2\beta_r^{k,N}\gamma_r^{k,N}z_r^{k,N}v + \beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2|r-r_k|^2 v + 2\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2\eta_r^{k,N}(r-r_k)\partial v + z_r^{k,N}\left(\gamma_r^{k,N}\right)^3\partial v$$
$$\left.+2\gamma_r^{k,N}\eta_r^{k,N}z_r^{k,N}(r-r_k)v\right\}\left(r, X_r^N\right)dr$$
$$+\left\{\left|\gamma_r^{k,N}\right|^2 z_r^{k,N}v + \beta_r^{k,N}\left(\gamma_r^{k,N}\right)^3\partial v + 2\beta_r^{k,N}\gamma_r^{k,N}\eta_r^{k,N}(r-r_k)v\right\}\left(r, X_r^N\right)dW_r.$$

Using this equation with $v = \frac{1}{2}\partial^3 u$, integrating between s and t_{k+1} and taking expectation we have

$$-\frac{1}{2}E\left[\beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2\partial^3 u\left(s, X_s^N\right)\right] = -\frac{1}{2}E\left[\beta_{t_{k+1}}^{k,N}\left|\gamma_{t_{k+1}}^{k,N}\right|^2\partial^3 u\left(t_{k+1}, X_{t_{k+1}}^N\right)\right] + R_7(s),$$

with $|R_7(s)| \le Ch$.

3.7. **Upper estimate of $\mathcal{J}_k^N(t)$.** We upper estimate the error $\tilde\phi_i(s) - \tilde\phi_i(t_k)$ where $\tilde\phi_i$ is one of the nine terms in the right hand side of (3.12)

3.7.1. *The term* $\tilde\phi_1(s) = \left|\gamma_s^{k,N}\right|^2\frac{\partial}{\partial t}\Delta u\left(s, X_s^N\right)$. Using (3.17) with $v = \frac{\partial}{\partial t}\Delta u$, integrating between t_k and s, taking expectation and using the fact that $\gamma_{t_k}^{k,N} = \sigma\left(X_{t_k}^N\right)$ we have

$$E\left[\left|\gamma_s^{k,N}\right|^2\frac{\partial}{\partial t}\Delta u\left(s, X_s^N\right)\right] = E\left[\left|\sigma\left(X_{t_k}^N\right)\right|^2\frac{\partial}{\partial t}\Delta u\left(t_k, X_{t_k}^N\right)\right] + \tilde R_1(s),$$

with

$$\tilde R_1(s) = E\int_{t_k}^s\left[\left|\gamma_r^{k,N}\right|^2\frac{\partial^2}{\partial r^2}\Delta u + \{2\gamma_r^{k,N}z_r^{k,N} + \left|\eta_r^{k,N}\right|^2|r-t_k|^2\}\frac{\partial}{\partial r}\Delta u\right.$$
$$\left.+\{\beta_r^{k,N} + 2\eta_r^{k,N}(r-t_k)\}\left|\gamma_r^{k,N}\right|^2\frac{\partial}{\partial r}\partial^3 u + \frac{1}{2}\left|\gamma_r^{k,N}\right|^4\frac{\partial}{\partial r}\partial^4 u\right]\left(r, X_r^N\right)dr.$$

Corollary 3.15 implies that $|\tilde R_1(s)| \le Ch$.

3.7.2. *The term* $\tilde{\phi}_2(s) = \beta_s^{k,N} \left|\gamma_s^{k,N}\right|^2 \partial^3 u\left(s, X_s^N\right)$. For an $\mathcal{F}_s$-measurable random variable Z, we have $E\left(Z\beta_{t_k}^{k,N}\right) = E\left(Zb\left(X_{t_{k+1}}^N\right)\right)$. Using (3.20) with $v = \partial^3 u$, integrating between t_k and s and taking expectation we have

$$E\left[\beta_s^{k,N}\left|\gamma_s^{k,N}\right|^2 \partial^3 u\left(s, X_s^N\right)\right] = E\left[b\left(X_{t_{k+1}}^N\right)\left|\sigma\left(X_{t_k}^N\right)\right|^2 \partial^3 u\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_2(s),$$

where

$$\begin{aligned}\tilde{R}_2(s) := E\int_{t_k}^s \Big\{ &\{2\gamma_r^{k,N} z_r^{k,N}\beta_r^{k,N} + \beta_r^{k,N}\left|\eta_r^{k,N}\right|^2 |r-t_k|^2 + 2\gamma_r^{k,N}\eta_r^{k,N} z_r^{k,N}(r-t_k)\}\partial^3 u \\ &+ \beta_r^{k,N}\left|\gamma_r^{k,N}\right|^2 \frac{\partial}{\partial r}\partial^3 u + \frac{1}{2}\beta_r^{k,N}\left|\gamma_r^{k,N}\right|^4 \partial^5 u \\ &+ \{\left|\beta_r^{k,N}\right|^2 + 2\beta_r^{k,N}\eta_r^{k,N}(r-t_k) + \gamma_r^{k,N} z_r^{k,N}\}\left|\gamma_r^{k,N}\right|^2 \partial^4 u\Big\}\left(r, X_r^N\right) dr.\end{aligned}$$

Corollary 3.15 implies that $|\tilde{R}_2(s)| \le Ch$.

3.7.3. *The term* $\tilde{\phi}_3(s) = \frac{1}{2}\left|\gamma_s^{k,N}\right|^4 \partial^4 u\left(s, X_s^N\right)$. Using Lemma 3.10 and Itô's formula we deduce

$$d\left|\gamma_r^{k,N}\right|^4 = \left|\gamma_r^{k,N}\right|^2 \{4\gamma_r^{k,N} z_t^{k,N} + 6\left|\eta_r^{k,N}\right|^2 |r-t_k|^2\}dr + 4\left(\gamma_r^{k,N}\right)^3 \eta_r^{k,N}(r-t_k)dW_r.$$

Using this equation and (3.13) with $v = \frac{1}{2}\partial^4 u$ and applying Itô formula, we have

$$\frac{1}{2}E\left[\left|\gamma_s^{k,N}\right|^4 \partial^4 u\left(s, X_s^N\right)\right] = \frac{1}{2}E\left[\left|\sigma\left(X_{t_k}^N\right)\right|^4 \partial^4 u\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_3(s),$$

where $\tilde{R}_3(s) \le Ch$ by Corollary 3.15.

3.7.4. *The term* $\tilde{\phi}_4(s) = 2\gamma_s^{k,N} z_s^{k,N}\Delta u\left(s, X_s^N\right)$. Using (3.19) with $v = 2\Delta u$ we have $E\left[2\gamma_s^{k,N} z_s^{k,N}\Delta u\left(s, X_s^N\right)\right] = E\left[2\gamma_{t_k}^{k,N} z_{t_k}\Delta u\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_4(s)$, and Corollary 3.15 implies $|\tilde{R}_4(s)| \le Ch$.

3.7.5. *The term* $\tilde{\phi}_5(s) := \tilde{R}_5(s) := |s-t_k|^2 \left|\eta_s^{k,N}\right|^2 \Delta u\left(s, X_s^N\right) + 2(s-t_k)\left|\gamma_s^{k,N}\right|^2 \eta_s^{k,N}\partial^3 u\left(s, X_s^N\right)$. Using Corollary 3.15, we have $|\tilde{R}_5(s)| \le Ch$.

3.7.6. *The term* $\tilde{\phi}_6(s) = \frac{\partial}{\partial t}\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)$. Using (3.13) with $v = \frac{\partial}{\partial t}\left(\sigma^2\Delta u\right)$, integrating between t_k and s and taking expectation, we have

$$-E\left[\frac{\partial}{\partial t}\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)\right] = -E\left[\frac{\partial}{\partial t}\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_6(s),$$

with $|\tilde{R}_6(s)| \le Ch$ by Corollary 3.15.

3.7.7. *The term* $\tilde{\phi}_7(s) = \beta_s^{k,N}\partial\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)$. Using (3.14) with $v = \partial(\sigma^2\Delta u)$, integrating between t_k and s, taking expectation we have

$$-E\left[\beta_s^{k,N}\partial\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)\right] = -E\left[b\left(X_{t_k}^N\right)\partial\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_7(s),$$

with $|\tilde{R}_7(s)| \le Ch$ by Corollary 3.15.

3.7.8. *The term* $\tilde{\phi}_8(s) = \frac{1}{2}\left|\gamma_s^{k,N}\right|^2 \Delta\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)$. Using (3.17) with $v = \frac{1}{2}\Delta(\sigma^2\Delta u)$, integrating between t_k and s, and finally taking expectation we have

$$-\frac{1}{2}E\left[\left|\gamma_s^{k,N}\right|^2 \Delta\left(\sigma^2\Delta u\right)\left(s, X_s^N\right)\right] = -\frac{1}{2}E\left[\left|\sigma\left(X_{t_k}^N\right)\right|^2 \Delta\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right)\right] + \tilde{R}_8(s),$$

with $|\tilde{R}_8(s)| \le Ch$ by Corollary 3.15.

3.8. **Proof Theorem 2.1 (i).** The identity (3.18) and the upper estimate in section 3.6 and 3.7 imply that

$$\begin{aligned} Ef\left(X_T^N\right) - Ef\left(X_T\right) =& \sum_{k=0}^{N-1} E\int_{t_k}^{t_{k+1}} \left\{\int_t^{t_{k+1}} i_k^N(t_{k+1})ds + \frac{1}{2}\int_{t_k}^t j_k^N(t_k)ds\right\} dt + R \\ =& \frac{1}{2}h^2\sum_{k=0}^{N-1} Ei_k^N(t_{k+1}) + \frac{1}{4}h^2\sum_{k=0}^{N-1} Ej_k^N(t_k) + R, \end{aligned} \tag{3.21}$$

where

$$R := \sum_{k=0}^{N-1}\sum_{j=1}^{7}\int_{t_k}^{t_{k+1}}\int_t^{t_{k+1}} R_j(s)dsdt + \sum_{k=0}^{N-1}\sum_{j=1}^{8}\int_{t_k}^{t_{k+1}}\int_{t_k}^t \tilde{R}_j(s)dsdt.$$

Hence $|R| \le Ch^2$. Note that $\beta_{t_{k+1}}^{k,N} = b\left(X_{t_{k+1}}^N\right)$. Using (3.3) and (3.5) we deduce that $z_{t_{k+1}}^{k,N} = \sigma\left(X_{t_k}^N\right)(S_h b')\left(X_{t_{k+1}}^N\right)$ and $\gamma_{t_{k+1}}^{k,N} = \sigma\left(X_{t_k}^N\right)\left[1 + h(S_h b')\left(X_{t_{k+1}}^N\right)\right] = \sigma\left(X_{t_k}^N\right)S_h\left(X_{t_{k+1}}^N\right)$. Therefore, we deduce that

$$\begin{aligned} i_k^N(t_{k+1}) =& \frac{\partial}{\partial t}(b\partial u)\left(t_{k+1}, X_{t_{k+1}}^N\right) + \left[b\partial(b\partial u)\right]\left(t_{k+1}, X_{t_{k+1}}^N\right) + \frac{1}{2}\sigma^2\left(X_{t_k}^N\right)\left[S_h^2\Delta(b\partial u)\right]\left(t_{k+1}, X_{t_{k+1}}^N\right) \\ &- \left[b\frac{\partial}{\partial t}\partial u\right]\left(t_{k+1}, X_{t_{k+1}}^N\right) - \left[b^2\Delta u\right]\left(t_{k+1}, X_{t_{k+1}}^N\right) - \sigma^2\left(X_{t_k}^N\right)\left[b' S_h^2\Delta u\right]\left(t_{k+1}, X_{t_{k+1}}^N\right) \\ &- \frac{1}{2}\sigma^2\left(X_{t_k}^N\right)\left[bS_h^2\partial^3 u\right]\left(t_{k+1}, X_{t_{k+1}}^N\right). \end{aligned}$$

Similary, $\gamma_{t_k}^{k,N} = \sigma\left(X_{t_k}^N\right)$. So we have

$$\begin{aligned} Ej_k^N(t_k) =& E\sigma^2\frac{\partial}{\partial t}\Delta u\left(t_k, X_{t_k}^N\right) + Eb\left(X_{t_{k+1}}^N\right)\sigma^2\partial^3 u\left(t_k, X_{t_k}^N\right) + \frac{1}{2}E\sigma^4\partial^4 u\left(t_k, X_{t_k}^N\right) \\ &+ 2ES_h b'\left(X_{t_{k+1}}^N\right)\sigma^2\Delta u\left(t_k, X_{t_k}^N\right) - E\frac{\partial}{\partial t}\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right) \\ &- Eb\left(X_{t_{k+1}}^N\right)\partial\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right) - \frac{1}{2}E\sigma^2\Delta\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right). \end{aligned}$$

Notice that b and σ do not depend upon t; hence after simplification we have

$$i_k^N(t_{k+1}) = b\partial(b\partial u)\left(t_{k+1}, X_{t_{k+1}}^N\right) - b^2\Delta u\left(t_{k+1}, X_{t_{k+1}}^N\right) + \frac{1}{2}\sigma^2\left(X_{t_k}^N\right)\left(S_h^2 b''\partial u\right)\left(t_{k+1}, X_{t_{k+1}}^N\right), \tag{3.22}$$

$$\begin{aligned} Ej_k^N(t_k) =& Eb\left(X_{t_{k+1}}^N\right)\sigma^2\partial^3 u\left(t_k, X_{t_k}^N\right) + \frac{1}{2}E\sigma^4\partial^4 u\left(t_k, X_{t_k}^N\right) - Eb\left(X_{t_{k+1}}^N\right)\partial\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right) \\ &- \frac{1}{2}E\sigma^2\Delta\left(\sigma^2\Delta u\right)\left(t_k, X_{t_k}^N\right) + 2Eb' S_h\left(X_{t_{k+1}}^N\right)\sigma^2\Delta u\left(t_k, X_{t_k}^N\right). \end{aligned} \tag{3.23}$$

Corollary 3.15 implies the existence of a constant C, such that for all $k = 0, \ldots, N-1$:

$$\left|E\left(i_k^N(t_{k+1}) + j_k^N(t_k)\right)\right| \le C.$$

Using this bound with (3.21) proves the first part of Theorem 2.1.

3.9. **Proof Theorem 2.1 (ii).** We at first prove the following lemma, which upper estimates the error in the approximation of an integral by a Riemann sum.

Lemma 3.18. *Let v and w in $C_b^{\infty,\infty}([0,T]\times\mathbf{R})$. Then there exists a constant C independent of h such that*

$$\left|h\sum_{k=0}^{N-1} Ev\left(t_{k+1},X^N_{t_{k+1}}\right)w\left(t_k,X^N_{t_k}\right)-E\int_0^T vw(t,X_t)dt\right|\le Ch.$$

Proof. Using (3.13) multiply by $w(t_k,X^N_{t_k})$ and taking expected value, we deduce for $v\in C^{1,2}$, $Ev\left(t_{k+1},X^N_{t_{k+1}}\right)w\left(t_k,X^N_{t_k}\right)=Evw\left(t_k,X^N_{t_k}\right)+A_k$, where

$$A_k:=Ew\left(t_k,X^N_{t_k}\right)\int_{t_k}^{t_{k+1}}\left\{\frac{\partial}{\partial t}v+\beta_t^{k,N}\partial v+\frac{1}{2}\left|\gamma_t^{k,N}\right|^2\Delta v\right\}\left(t,X_t^N\right)dt.$$

This yields

$$h\sum_{k=0}^{N-1} Ev\left(t_{k+1},X^N_{t_{k+1}}\right)w\left(t_k,X^N_{t_k}\right)-E\int_0^T vw(t,X_t)dt=\sum_{k=0}^{N-1}(hA_k+hB_k+C_k).$$

where

$$\begin{aligned}B_k:=&E(vw)\left(t_k,X^N_{t_k}\right)-E(vw)\left(t_k,X_{t_k}\right)\\ C_k:=&hE(vw)\left(t_k,X_{t_k}\right)-E\int_{t_k}^{t_{k+1}}(vw)(t,X_t)dt.\end{aligned}$$

Using the Cauchy-Schwarz inequality, the fact that $\frac{\partial}{\partial t}v$, ∂v and Δv have polynomial grow so that Corollary 3.15 can be applied, we deduce

$$\begin{aligned}|A_k|^2\le&E\left[\left|w\left(t_k,X^N_{t_k}\right)\right|^2\right]E\left[\left|\int_{t_k}^{t_{k+1}}\left\{\frac{\partial}{\partial t}v+\beta_t^{k,N}\partial v+\frac{1}{2}\left|\gamma_t^{k,N}\right|^2\Delta v\right\}\left(t,X_t^N\right)dt\right|^2\right]\\ \le&ChE\int_{t_k}^{t_{k+1}}\left|\left\{\frac{\partial}{\partial t}v+\beta_t^{k,N}\partial v+\frac{1}{2}\left|\gamma_t^{k,N}\right|^2\Delta v\right\}\left(t,X_t^N\right)\right|^2dt\\ \le&Ch^2,\end{aligned}$$

Hence, $|A_k|\le Ch$ which implies $h\sum_{0\le k\le N-1}|A_k|\le Ch$.
Since $(vw)(t_k,.)$ is in C_b^∞, we use Theorem 2.1 (i), changing T by t_k, which yields $|B_k|\le Ch$ and then $h\sum_{0\le k\le N-1}|B_k|\le Ch$. Finally, Itô's formula implies

$$\begin{aligned}C_k=&E\int_{t_k}^{t_{k+1}}\left\{(vw)\left(t_k,X_{t_k}\right)-(vw)\left(t,X_t\right)\right\}dt\\ =&-E\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\left\{\frac{\partial}{\partial t}(vw)+b\partial(vw)+\frac{1}{2}\sigma^2\Delta(vw)\right\}(s,X_s)dsdt.\end{aligned}$$

Once more the polynomial growth imposed on v, w and their partial derivatives implies that $|C_k|\le Ch^2$ and then $\sum_{0\le k\le N-1}|C_k|\le Ch$. This concludes the proof. □

Now we introduce the function $\psi_{ih}:[0,T]\times\mathbf{R}\to\mathbf{R}$ defined by

$$\begin{aligned}\psi_{ih}(t,x):=&\frac{1}{2}b\partial(b\partial u)(t,x)-\frac{1}{2}b^2\Delta u(t,x)+\frac{1}{4}\sigma^2S_h^2b''\partial u(t,x)+\frac{1}{4}b\sigma^2\partial^3u(t,x)\\ &+\frac{1}{8}\sigma^4\partial^4u(t,x)+\frac{1}{2}b'S_h\sigma^2\Delta u(t,x)-\frac{1}{4}b\partial(\sigma^2\Delta u)(t,x)-\frac{1}{8}\sigma^2\Delta(\sigma^2\Delta u)(t,x)\end{aligned}\tag{3.24}$$

Using the expression of i_k^N in (3.22) (resp. j_k^N in (3.23)) and the previous lemma, we deduce

$$\left|\frac{1}{2}h\sum_{k=0}^{N-1}Ei_k^N(t_{k+1})+\frac{1}{4}h\sum_{k=0}^{N_1}Ej_k^N(t_k)-\int_0^T E\psi_{ih}(t,X_t)dt\right|\le Ch. \tag{3.25}$$

Using the definitions of ψ_i and ψ_{ih} given in (2.2) and (3.24) respectively, we have

$$\begin{aligned}\psi_{ih}(t,x)-\psi_i(t,x)=&\frac{1}{4}\left\{\sigma^2 S_h^2 b''\partial u+b\sigma^2\partial^3 u+2b'S_h\sigma^2\Delta u-\sigma^2\Delta(b\partial u)\right\}(t,x)\\ =&\frac{1}{4}\sigma^2\left(S_h^2-1\right)b''\partial u(t,x)+\frac{1}{2}b'\left(S_h-1\right)\sigma^2\Delta u(t,x).\end{aligned}$$

Since $\left(S_h-1\right)(x)=\frac{hb'}{1-hb'}(x)$ and $|S_h(x)+1|\le C$ for $h\in(0,h^*)$, we have $|\left(S_h-1\right)(x)|+\left|\left(S_h^2-1\right)(x)\right|\le Ch$, where as usually C does not depend on N and h. This yields

$$\left|\int_0^T E\left\{\psi_{ih}(t,X_t)-\psi_i(t,X_t)\right\}dt\right|\le Ch.$$

This last equation with (3.21) and (3.25) concludes the proof.

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References

[1] Bally V., Talay D., The law of the Euler scheme for stochastic differential equations: I. Convergence rate of the distribution function. *Prob. Th. Rel. Fields*, 104-1:43–60, 1996.

[2] de Bouard A., Debussche A., Weak and strong order of convergence of a semi discrete scheme for the stochastic Nonlinear Schrodinger equation, *Applied Mathematics and Optimization journal*, 54, 3, pp. 369–399 2006.

[3] Debussche A., Weak approximation of stochastic partial differential equations: the nonlinear case, *Math of Comp*, 80, pp. 89–117 2011.

[4] Debussche A., Printems J., Weak order for the discretization of the stochastic heat equation, *Math of Comp*, 78, 266, pp. 845–863 2009.

[5] Kloeden P. E., Platen E., *Numerical solution of stochastic differential equations.* Applications of Mathematics (New York), 23. *Springer-Verlag, Berlin*, 1992.

[6] Nualart D., *The Malliavin Calculus and Related Topics. Second Edition.* Probability and its Applications (New York). *Springer-Verlag, Berlin,* 2006.

[7] Talay D., Tubaro L., Expansion of the global error for numerical schemes solving SDEs *Stoch. Anal. and App.*, 8(4),483–509, 1990.

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